A note on relaxations of the choice network revenue management dynamic program

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Abstract

In recent years, a number of approximation methods have been proposed for the choice network revenue management problem. These approximation methods are motivated by the fact that the dynamic programming formulation of the choice network revenue management problem is intractable even for moderately sized instances. In this paper, we consider three approximation methods that obtain upper bounds on the value function, namely the choice deterministic linear program (CDLP), the affine approximation (AF) and the piecewise-linear approximation (PL). It is known that the piecewise-linear approximation bound is tighter than the affine bound, which in turn is tighter than CDLP. In this paper, we prove bounds on how much the affine and piecewise-linear approximations can tighten CDLP. We show (i) the gap between the AF and CDLP bounds is at most a factor of $1 + \frac{1}{\min_i r_i}$, where $r_i > 0$ are the resource capacities, and (ii) the gap between the piecewise-linear and CDLP bounds is within a factor of 2. Moreover we show that these gaps are essentially tight. Our results hold for any discrete-choice model and do not involve any asymptotic scaling. Our results are surprising as calculating the AF bound is NP-hard and CDLP is tractable for a single-segment MNL; our result implies that if a firm has all resource capacities of 100, the gap between the two bounds however is at most 1.01.

1 Introduction and literature review

Network revenue management (NRM) is the problem of maximizing the sale of a set of resources (the network) by creating differentiated products at different prices and controlling the sale of the products. The purchasing decisions of the customers are influenced by the assortment, or the set of products, made available for sale. Moreover, the products consume different resources and there are limited quantities of the resources available. Therefore, the decision on what set of products to make available for sale over time has to factor in the resource availabilities and the underlying model of customer choice.

The NRM model has a number of applications including the airline, car rental, display advertising and hotel industries; see Talluri and van Ryzin [12]. While the choice NRM problem can be

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formulated as a stochastic dynamic program, it turns out to be intractable even for moderately sized instances and simple models of choice.

This has motivated a number of approximation methods in the literature. The choice deterministic linear program (CDLP) proposed by Liu and van Ryzin [7] and Gallego, Iyengar, Phillips, and Dubey [3] is one widely used approximation method. Liu and van Ryzin [7] show that the optimal objective function value of CDLP is an upper bound on the value function and describe different ways in which the CDLP solution can be used to obtain various control policies. Liu and van Ryzin [7] also describe a dynamic programming decomposition approach that uses the optimal dual solution to CDLP to decompose the network problem into a number of single resource problems.


While the stronger approximation methods improve the CDLP bound, they come with significant computational cost; see the numerical studies in Zhang and Adelman [14], Meissner and Strauss [8], Kunnumkal and Topaloglu [6] and Kunnumkal and Talluri [5]. In general, the tractability of the approximation methods depends on the underlying discrete-choice model. Liu and van Ryzin [7] show that CDLP is tractable for the MNL model as long as the consideration sets, the sets of products of interest to the different customer segments, are disjoint. Zhang and Adelman [14] use column generation and integer programming to solve AF. On the other hand, Kunnumkal and Talluri [5] show that the AF column generation is NP-hard even for the single-segment MNL model (this also implies a similar hardness result for the piecewise-linear approximation (PL) of Meissner and Strauss [8]; note that by Liu and van Ryzin [7] CDLP is solvable in polynomial-time for the MNL model).

Our result therefore is surprising: we show the gap between AF and CDLP is negligible except when the capacities are small: the CDLP bound is within a factor of $1 + \frac{1}{\min_i r^i}$ of AF, where $r^i > 0$ are the initial capacities of the resources. We then consider the gap between the PL and CDLP bounds and show that the gap is at most 2. In doing this, we also establish some useful structural properties of the solution to the PL linear program.

The approximation proposed by Kunnumkal and Talluri [5] is known to be weaker than AF and so its improvement over CDLP is also bounded by a factor of $1 + \frac{1}{\min_i r^i}$. The dynamic programming decomposition approach of Liu and van Ryzin [7] and the Lagrangian relaxation method of Kunnumkal and Topaloglu [6] obtain separable piecewise-linear approximations to the value function, and are known to be weaker than PL. Therefore, their improvements over CDLP are also bounded by a factor of 2. We emphasize that our results apply to a general discrete-choice model and do not rely on asymptotic scaling such as in Talluri and van Ryzin [11], Liu and van Ryzin [7] and Cooper [2]. Our results are surprising as calculating the AF bound is NP-hard and CDLP is tractable for a single-segment MNL; our result implies that if an airline say has flights where all the planes have a capacity of 100, the gap between the two however is at most 1.01.
To summarize, we make the following research contributions:

1. We establish limits on how much $AF$ and $PL$ can improve the $CDLP$ bound for any discrete-choice model and at all capacity and demand levels.

2. For $PL$ we establish new structural properties of the $PL$ linear program. These properties could be useful to speed up its solution time.

3. Finally, we show that the gaps are tight.

The remainder of the paper is organized as follows: In §2 we describe the choice NRM model, the notation, and the stochastic dynamic programming formulation of the choice NRM problem. In §3 we describe the choice deterministic linear program. In §4 we consider the affine approximation and establish the worst case gap between the affine and the $CDLP$ bounds. In §5, we consider the piecewise-linear approximation and show that the $CDLP$ bound is within a factor of 2 of the piecewise-linear approximation bound. We give an example in §6 which shows that the gaps are tight.

## 2 Problem formulation

Our model and notation is to a large part based on Liu and van Ryzin [7]. A product is a specification of a price and the set of resources that it consumes. For example, a product would be an itinerary-fare class combination for an airline network, where an itinerary is a combination of flight legs. On the other hand, in a hotel network, a product would be a multi-night stay for a particular room type at a certain price point. Time is discrete and the sales horizon is assumed to consist of $\tau$ intervals, indexed by $t$. The sales horizon begins at time $t = 1$ and ends at $t = \tau$; all the resources perish instantaneously at time $\tau + 1$. We make the standard assumption that the time intervals are fine enough so that the probability of more than one customer arriving in any single time period is negligible.

We let $I$ denote the set of resources and $J$ the set of products. We index resources by $i$ and products by $j$. We let $f_j$ denote the revenue associated with product $j$ and use $I_j \subseteq I$ to denote the set of resources used by product $j$. We use $J_i \subseteq J$ to denote the set of products that use resource $i$.

In each period the firm offers a subset $S$ of its products for sale, called the offer set. We write $i \in I_S$ whenever there is a $j \in S$ with $i \in I_j$. That is, there is at least one product in the offer set $S$ that uses resource $i$.

We use superscripts on vectors to index the vectors (for example, the resource capacity vector associated with time period $t$ would be $r^t$) and subscripts to indicate components (for example, the capacity on resource $i$ in time period $t$ would be $r^t_i$). Therefore, $r^1 = [r^1_i]$ represents the initial capacity on the resources and $r^t = [r^t_i]$ denotes the remaining capacity on the resources at the beginning of time period $t$. The remaining capacity $r^t_i$ takes values in the set $R_i = \{0, \ldots, r^1_i\}$ and $\mathcal{R} = \prod_i \mathcal{R}_i$ represents the state space. We use $\mathbb{I}_{\mathbb{C}}$ as the indicator function that takes 1 if the condition in $\{ \}$ is true and 0 if false.
2.1 Demand model

In each time period, a customer arrives with probability \( \alpha \). Given an offer set \( S \), an arriving customer purchases a product \( j \) in the set \( S \) or leaves without making a purchase. The no-purchase option is indexed by 0 and is always present for the customer. We let \( P_j(S) \) denote the probability that the firm sells product \( j \) given that a customer arrives and the offer set is \( S \). Clearly, \( P_j(S) = 0 \) if \( j \notin S \). The probability of no sale given a customer arrival is \( P_0(S) = 1 - \sum_{j \in S} P_j(S) \). We assume that the choice probabilities are given by an oracle, as the model represents a general discrete-choice model; they could conceivably be calculated by a simple formula as in the case of the MNL model.

We assume that the arrival rates and choice probabilities are stationary. This is for brevity of notation and all of our results go through with nonstationary arrival rates and choice probabilities.

2.2 Choice dynamic program

The dynamic program to determine optimal controls is as follows. Let \( V_t(r_t) \) denote the maximum expected revenue to go, given remaining capacity \( r_t \) at the beginning of period \( t \). Then \( V_t(r_t) \) must satisfy the Bellman equation

\[
V_t(r_t) = \max_{S \subseteq S(r_t)} \left\{ \sum_{j \in S} \alpha P_j(S) \left[ f_j + V_{t+1} \left( r_t - \sum_{i \in I_j} e^i \right) \right] + \left[ \alpha P_0(S) + 1 - \alpha \right] V_{t+1}(r_t) \right\},
\]

where

\[
S(r) = \{ j \mid \sum_{i \in I_j} r_i \leq r_t, \forall i \}
\]

represents the set of products that can be offered given the capacity vector \( r \) and \( e^i \) is a vector with a 1 in the \( i \)th position and 0 elsewhere. The boundary conditions are \( V_{t+1}(r) = V_t(0) = 0 \) for all \( r \) and for all \( t \), where 0 is a vector of all zeroes. \( V^{DP} = V_t(r_t) \) denotes the optimal expected total revenue over the booking horizon, given the initial capacity vector \( r^t \).

For brevity of notation, we assume that \( \alpha = 1 \) in the remaining part of the paper. We note that this is without loss of generality since it is equivalent to letting \( \bar{P}_j(S) = \alpha P_j(S) \) and \( \bar{P}_0(S) = \alpha P_0(S) + 1 - \alpha \), and working with the choice probabilities \( \{ \bar{P}_j(S) \mid \forall j, S \} \).

2.3 Linear programming formulation of the dynamic program

The value functions can, alternatively, be obtained by solving a linear program (LP), following Schweitzer and Seidmann [10]. The LP formulation of the choice NRM dynamic program has a decision variable \( V_t(r) \) for each state vector \( r \) in each period \( t \) and is as follows:

\[
V^{DP} = \min_V \left\{ V_t(r^t) \right\}
\]

s.t.

\[
(DPLP) \quad V_t(r) \geq \sum_j P_j(S) \left[ f_j + V_{t+1} \left( r - \sum_{i \in I_j} e^i \right) - V_{t+1}(r) \right] + V_{t+1}(r) \quad \forall r \in R, S \subseteq S(r), t
\]

\[
V_t(r) \geq 0 \quad \forall r, t
\]
with the boundary conditions that $V_{t+1}(r) = 0$ for all $r$ and $V_t(0) = 0$ for all $t$. Both dynamic program (1) and linear program $DPLP$ are computationally intractable, but linear program $DPLP$ turns out to be useful in developing value function approximation methods. In the following sections, we describe methods to approximate the value function.

3 Choice deterministic linear program

The choice deterministic linear program ($CDLP$) proposed in Gallego et al. [3] and Liu and van Ryzin [7] is given by

$$
V_{CDLP} = \max_h \sum_t \sum_S R(S) h_{S,t}
$$

s.t.

$$(CDLP) \quad \sum_{k=1}^t \sum_S Q_i(S) h_{S,k} \leq r_i^t \quad \forall i, t \quad (3)$$

$$\sum_S h_{S,t} = 1 \quad \forall t \quad (4)$$

$$h_{S,t} \geq 0 \quad \forall S, t
$$

where $R(S) = \sum_j P_j(S) f_j$ is the expected revenue obtained by offering set $S$ and $Q_i(S) = \sum_{j \in \mathcal{J}_i} P_j(S)$ is the expected capacity consumed on resource $i$ when $S$ is offered. In the above LP, we interpret the decision variable $h_{S,t}$ as the frequency with which set $S$ is offered at time period $t$. The objective function measures the total expected revenues, while the first set of constraints ensure that the total expected capacity consumed on each resource up until time period $t$ does not exceed its available capacity. Note that since $h_{S,t} \geq 0$, constraints (3) are redundant except for the last time period. Still, this expanded formulation is useful when we compare $CDLP$ with other approximation methods. The second set of constraints ensures that the total frequencies add up to 1.

Liu and van Ryzin [7] show that the optimal objective function value of $CDLP$ gives an upper bound on the optimal expected revenue. That is, $V_{DP} \leq V_{CDLP}$. Since $CDLP$ has an exponential number of decision variables it has to be solved using column generation. Liu and van Ryzin [7] show that the $CDLP$ column generation can be carried out efficiently provided demand is comprised of multiple customer segments with disjoint consideration sets and choice within each segment is according to the MNL model. The column generation procedure is intractable in general; if the consideration sets overlap, then it is known to be NP-complete even for the MNL model with just two segments (Bront, Méndez-Díaz, and Vulcano [1] and Rusmevichientong, Shmoys, Tong, and Topaloglu [9]).

The dual formulation of $CDLP$ is useful for bounding its gap with the affine and piecewise-linear approximations. Associating dual variables $\gamma = \{\gamma_{i,t} | \forall i, t\}$ with constraints (3) and $\beta = \{\beta_t | \forall t\}$ with constraints (4), the dual of $CDLP$ is

$$
V_{CDLP}^{d\gamma} = \min_{\beta, \gamma} \sum_t \beta_t + \sum_t \sum_i \gamma_{i,t} r_i^t
$$

s.t.

$$(dCDLP) \quad \beta_t + \sum_{k=t}^t \left(\sum_{k=t}^t \gamma_{i,k}\right) Q_i(S) \geq R(S) \quad \forall t, S \quad (5)$$

$$\gamma_{i,t} \geq 0 \quad \forall i, t
$$
4 Affine approximation

We show in this section that the gap between $AF$ and $CDLP$ becomes negligibly small as the resource capacities get large. In particular, the $CDLP$ bound is within a factor of $1 + \frac{1}{\min_i \{r_i\}}$ of the affine bound for any discrete-choice model.

Zhang and Adelman [14] propose replacing the value function $V_t(r)$ by the affine function $\theta_t + \sum_i V_i,t r_i$ in $DPLP$ to obtain the affine approximation ($AF$) linear program

$$V^{AF} = \min_{\theta, V} \theta_1 + \sum_i V_i,1 r_i^1$$

s.t.

$$\theta_i + \sum_i V_i,t r_i \geq \sum_j P_j(S) \left[ f_j - \sum_{i \in I_j} V_i,t+1 \right] + \theta_{t+1} + \sum_i V_i,t+1 r_i$$

$$\forall r \in R, S \subseteq S(r), t$$

$$\theta_t, V_i,t \geq 0 \quad \forall i, t.$$ 

Zhang and Adelman [14] show that $V^{AF}$ is an upper bound on the optimal expected revenue and this bound is tighter than the $CDLP$ upper bound. That is, $V^{AF} \leq V^{CDLP}$.

While the number of decision variables in $AF$ is manageable, the number of constraints is exponential both in the number of resources as well as the products. Vossen and Zhang [13] show that $AF$ has the following, equivalent, reduced formulation where the number of constraints is exponential only in the number of products:

$$V^{AF} = \min_{\beta, \gamma} \sum_t \beta_t + \sum_i \sum_k \gamma_i,k r_i$$

s.t.

$$\beta_t + \sum_i \sum_k [\sum_{k=t+1}^\tau \gamma_i,k] Q_i(S) \geq R(S) \quad \forall t, S$$

$$\gamma_i,t \geq 0 \quad \forall i, t;$$

we refer the reader to Vossen and Zhang [13] for details. While the number of constraints in $RAF$ is an order of magnitude smaller than $AF$, it is still exponential in the number of products and $RAF$ has to be solved by constraint generation. Kunnumkal and Talluri [5] show that the separation problem is intractable even for the MNL choice model with a single customer segment.

Given the computational complexity of $AF$, it is important to understand by how much it can tighten the $CDLP$ bound. Proposition 1 below characterizes the gap between the $AF$ and $CDLP$ bounds.

**Proposition 1.** $V^{CDLP} \leq \left( 1 + \frac{1}{\min_i \{r_i\}} \right) V^{AF}$.

**Proof.** Let $(\hat{\beta}, \hat{\gamma})$ be an optimal solution to $RAF$. Constraints (6) imply that

$$\hat{\beta}_t \geq \max_S \left\{ R(S) - \sum_i \left[ \sum_{k=t+1}^\tau \hat{\gamma}_i,k \right] Q_i(S) - \sum_i \sum_{i \in I_S} \hat{\gamma}_i,t \right\}.$$
Adding $\sum_i \hat{\gamma}_{i,t}$ to both sides of the above inequality, we have
\[
\hat{\beta}_t + \sum_t \hat{\gamma}_{i,t} \geq \max_S \left\{ R(S) - \sum_i \left( \sum_{k=t+1}^\tau \hat{\gamma}_{i,k} \right) Q_i(S) + \sum_t \hat{\gamma}_{i,t} \left( 1 - \mathbb{I}_{i \in I_S} \right) \right\}
\]
\[
\geq \max_S \left\{ R(S) - \sum_i \sum_{k=t}^\tau Q_i(S) \hat{\gamma}_{i,k} \right\}
\]
where the last inequality holds since $\hat{\gamma}_{i,t} \geq 0$ and $1 - \mathbb{I}_{i \in I_S} \geq 0$. Letting $\tilde{\beta}_t = \hat{\beta}_t + \sum_t \hat{\gamma}_{i,t}$, it follows that $(\tilde{\beta}, \tilde{\gamma})$ is a feasible solution to $dCDLP$ since it satisfies constraints (5). It follows that
\[
V^{CDLP} \leq \sum_t \tilde{\beta}_t + \sum_t \sum_i \tilde{\gamma}_{i,t} r_i^1
\]
\[
= \sum_t \left( \tilde{\beta}_t + \sum_i \tilde{\gamma}_{i,t} \right) + \sum_t \sum_i \tilde{\gamma}_{i,t} r_i^1
\]
\[
= \sum_t \tilde{\beta}_t + \sum_t \sum_i \tilde{\gamma}_{i,t} (1 + r_i^1)
\]
\[
\leq \sum_t \tilde{\beta}_t \left( 1 + \frac{1}{\min_v r_v^1} \right) + \sum_t \sum_i \tilde{\gamma}_{i,t} \left( \frac{r_i^1}{\min_v r_v^1} + r_i^1 \right)
\]
\[
= \left( 1 + \frac{1}{\min_v r_v^1} \right) \left[ \sum_t \tilde{\beta}_t + \sum_t \sum_i \tilde{\gamma}_{i,t} r_i^1 \right]
\]
\[
= \left( 1 + \frac{1}{\min_v r_v^1} \right) V^{AF},
\]
where the second inequality uses the fact that $r_i^1 / \min_v r_v^1 \geq 1$. \(\square\)

Note that when $\min_i r_i^1 = 1$, $CDLP$ is within a factor of 2 of $AF$. For typical values of the $r_i^1$'s, say in the airline industry where they represent airplane capacities, $CDLP$ will be quite close to $AF$ at the beginning of the booking horizon. So, real improvements are only possible at low resource capacities, which typically occur towards the end of the booking horizon.

5 Piecewise-linear approximation

In this section, we consider the piecewise-linear approximation. We first establish new structural properties of the solution to the piecewise-linear approximation LP and then use these properties to show that the $CDLP$ bound is within a factor of 2 of the piecewise-linear bound.

Meissner and Strauss [8] propose approximating the value function $V_t(r)$ by $\sum_i v_{i,t}(r_i)$ in $DLP$ to obtain the piecewise-linear approximation $(PL)$ linear program:
\[
V^{PL} = \min \sum v_{i,1}(r_i)
\]
\[
\text{s.t.}
\]
\[
(PL) \quad \sum_i v_{i,t}(r_i) \geq \sum_j P_j(S) \left[ f_j + \sum_{i \in I_j} (v_{i,t+1}(r_i) - v_{i,t}(r_i)) \right] + \sum_i v_{i,t+1}(r_i)
\]
\[
\forall r \in \mathcal{R}, S \subseteq S(r), t
\]
with the boundary conditions \( v_{i,r+1}(r_i) = 0 \) for all \( i \) and \( r_i \) and \( v_{i,t}(0) = 0 \) for all \( i \) and \( t \). Since PL uses a more refined approximation architecture than AF, it is natural to expect that it obtains a tighter upper bound on the value function. Indeed, Meissner and Strauss [8] show \( V^{\text{DP}} \leq V^{\text{PL}} \leq V^{\text{AF}} \).

Lemma 1 below shows that an optimal solution to PL satisfies certain monotonicity properties. If we interpret \( v_{i,t}(r_i) \) as the value of having \( r_i \) units of resource \( i \) at time period \( t \), then \( v_{i,t}(r_i) - v_{i,t}(r_i - 1) \) can be interpreted as the marginal value of the \( r_i \)th unit of the resource at time period \( t \). Part (i) of the lemma shows that the marginal value of capacity is decreasing in \( t \) keeping \( r_i \) constant; part (ii) of the lemma shows that the marginal value of capacity is decreasing in \( r_i \) for a fixed \( t \); parts (iii) and (iv) show that the value of capacity is increasing in \( r_i \) and decreasing in \( t \).

**Lemma 1.** There exists an optimal solution \( \hat{v} = \{\hat{v}_{i,t}(r_i) \mid \forall t, i, r_i \in \mathcal{R}_i\} \) to PL such that

(i) \( \hat{v}_{i,t}(r_i) - \hat{v}_{i,t-1}(r_i) \geq \hat{v}_{i,t+1}(r_i) - \hat{v}_{i,t+1}(r_i - 1) \) for all \( t, i \) and \( r_i \in \mathcal{R}_i \setminus \{0\} \).

(ii) \( \hat{v}_{i,t}(r_i) - \hat{v}_{i,t-1}(r_i - 1) \geq \hat{v}_{i,t}(r_i - 1) - \hat{v}_{i,t}(r_i) \) for all \( t, i \) and \( r_i \in \mathcal{R}_i \setminus \{0, r_i\} \).

(iii) \( \hat{v}_{i,t}(r_i) \geq \hat{v}_{i,t+1}(r_i) \) for all \( t, i \) and \( r_i \in \mathcal{R}_i \).  

(iv) \( \hat{v}_{i,t}(r_i) \geq \hat{v}_{i,t+1}(r_i) \) for all \( t, i \) and \( r_i \in \mathcal{R}_i \).

**Proof.** Appendix.

The monotonicity properties described in Lemma 1 are intuitive and are satisfied by the single resource revenue management problem as well as the piecewise-linear approximation to the NRM problem with independent demands (Talluri and van Ryzin [12] and Kunnumkal and Talluri [4]). So it is reassuring that the monotonicity properties continue to hold for an approximation to the NRM dynamic program under a general discrete-choice model.

The properties described in Lemma 1 are helpful in showing that the CDLP bound is no more than twice the PL bound. They could also be of independent interest, since by Lemma 1, we can add the constraints described in parts (i)-(iv) of the lemma to PL without affecting its optimal objective function value. This can potentially speed up its solution time (Zhang and Adelman [14]).

Solving PL is at least as hard as solving AF and so it is important to understand by how much it can tighten the CDLP bound. Proposition 2 below characterizes the gap between the PL and CDLP bounds.

**Proposition 2.** \( V^{\text{CDLP}} \leq 2V^{\text{PL}} \).

**Proof.** We assume that without loss of generality that \( r_i^1 > 0 \) for all \( i \) so that \( S(r^1) = \mathcal{J} \). Let \( \hat{v} = \{\hat{v}_{i,t}(r_i) \mid \forall t, i, r_i \in \mathcal{R}_i\} \) denote an optimal solution to PL. We construct a feasible solution to dCDLP in the following manner. We set \( \hat{\beta}_t = \sum [\hat{v}_{i,t}(r_i^1) - \hat{v}_{i,t+1}(r_i^1)] \), \( \bar{\gamma}_t = \hat{v}_{i,1}(r_i^1) - \hat{v}_{i,1}(r_i^1 - 1) \) and \( \bar{\gamma}_t = 0 \) for \( t < \tau \). We note that \( \bar{\gamma}_t \geq 0 \) by part (iii) of Lemma 1. For \( S \subseteq \mathcal{J} = S(r^1) \), we have

\[
\hat{\beta}_t = \sum_i [\hat{v}_{i,t}(r_i^1) - \hat{v}_{i,t+1}(r_i^1)] \\
\geq R(S) - \sum_i Q_i(S)[\hat{v}_{i,t+1}(r_i^1) - \hat{v}_{i,t+1}(r_i^1 - 1)] \\
\geq R(S) - \sum_i Q_i(S)[\hat{v}_{i,1}(r_i^1) - \hat{v}_{i,1}(r_i^1 - 1)] \\
= R(S) - \sum_i Q_i(S)\left(\sum_{k=t}^{\tau} \bar{\gamma}_{i,k}\right)
\]
Table 1: Choice probabilities, expected resource consumptions and expected revenues for the single resource revenue management example.

<table>
<thead>
<tr>
<th>S</th>
<th>$P_{1,t}(S)$</th>
<th>$P_{2,t}(S)$</th>
<th>$Q_{1,t}(S)$</th>
<th>$R_t(S)$</th>
<th>$P_{1,t}(S)$</th>
<th>$P_{2,t}(S)$</th>
<th>$Q_{1,t}(S)$</th>
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<td>1</td>
<td>1</td>
<td>1</td>
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</tbody>
</table>

where the first inequality uses the fact that $\hat{v}$ satisfies constraint (7). The last inequality follows from part (i) of Lemma 1, which implies that $\hat{v}_{i,1}(r^1_i) - \hat{v}_{i,1}(r^1_i - 1) \geq \hat{v}_{i,t+1}(r^1_i) - \hat{v}_{i,t+1}(r^1_i - 1)$. The last equality follows from the definition of $\hat{\gamma}_{i,t}$.

Therefore, $(\hat{\beta}, \hat{\gamma})$ is feasible to $dCDLP$ and

$$V^{CDLP} \leq \sum_t \hat{\beta}_t + \sum_i \sum_t \hat{\gamma}_{i,t}r^1_i = \sum_i \hat{v}_{i,1}(r^1_i) + \sum_i r^1_i[\hat{v}_{i,1}(r^1_i) - \hat{v}_{i,1}(r^1_i - 1)].$$

(8)

On the other hand, part (ii) of Lemma 1 implies that $\hat{v}_{i,t}(\cdot)$ is concave. As a result

$$\hat{v}_{i,1}(r^1_i - 1) \geq \frac{1}{r^1_i} \hat{v}_{i,1}(0) + \frac{r^1_i - 1}{r^1_i} \hat{v}_{i,1}(r^1_i) \geq \hat{v}_{i,1}(r^1_i) - \frac{1}{r^1_i} \hat{v}_{i,1}(r^1_i),$$

where the last inequality holds since $\hat{v}_{i,t}(0) \geq 0$ (Lemma 1, part (iv)). The above chain of inequalities imply that $\hat{v}_{i,1}(r^1_i) \geq \sum_i r^1_i[\hat{v}_{i,1}(r^1_i) - \hat{v}_{i,1}(r^1_i - 1)]$. Using this in (8), we have $V^{CDLP} \leq 2 \sum_i \hat{v}_{i,1}(r^1_i) = 2V^{PL}$.

We note that Proposition 2 is not an asymptotic-type relation, and does not require any demand or capacity scaling. Moreover, it holds for a general discrete-choice model.

6 Tightness of the bounds

We give an example which illustrates that the gaps in Propositions 1 and 2 are essentially tight. Consider a revenue management problem involving a single resource and two products. We have a single unit of the resource ($r^1_i = 1$) and a sales horizon of length $\tau = 2$. Note that since we have a single unit of capacity on the resource, both $AF$ and $PL$ coincide in this example. So it is enough to show tightness for Proposition 1; that is the gap between $AF$ and $CDLP$ is 2.

We let $f_1 = 1/\epsilon$ and $f_2 = 1$, where $\epsilon > 0$. The choice probabilities, expected resource consumptions and expected revenues associated with the different offer sets are given in Table 1. Note that in Table 1, we use $P_{1,t}(S), Q_{1,t}(S)$ and $R_t(S)$ to indicate the choice probabilities, expected resource consumptions and expected revenues at time period $t$. It can be verified that the $RAF$ can be reduced to

$$V^{AF} = \min_{\beta_1, \beta_2, \gamma_{1,1}, \gamma_{1,2}} \beta_1 + \beta_2 + \gamma_{1,1} + \gamma_{1,2}
$$

s.t.

$$\beta_1 + \gamma_{1,1} + \gamma_{1,2} \geq 1$$

$$\beta_2 + \gamma_{1,2} \geq 1$$

$$\beta_1, \beta_2, \gamma_{1,1}, \gamma_{1,2} \geq 0$$

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and that $V^{AF} = 1$. On the other hand, it can be verified that $CDLP$ reduces to

$$V^{CDLP} = \min_{\beta_1, \beta_2, \gamma_{1,1}, \gamma_{1,2}} \beta_1 + \beta_2 + \gamma_{1,1} + \gamma_{1,2}$$

s.t.

$$\beta_1 + \gamma_{1,1} + \gamma_{1,2} \geq 1$$

$$\beta_2 + \epsilon \gamma_{1,2} \geq 1$$

$$\beta_1, \beta_2, \gamma_{1,1}, \gamma_{1,2} \geq 0$$

and that $V^{CDLP} = 2 - \epsilon$.

As $\epsilon$ approaches zero, the ratio of $V^{AF}$ to $V^{CDLP}$ approaches 2.

References


Appendix

Proof of Lemma 1

We introduce some notation to simplify the expressions. Fixing a resource $l$, we let $\mathcal{R}_l(r_l) = \{x \in \mathcal{R} \mid x_l = r_l\}$ be the set of capacity vectors where the capacity on resource $l$ is fixed at $r_l$. Given a separable piecewise-linear approximation $v = \{v_{i,t}(r_i) \mid \forall t, i, r_i \in \mathcal{R}_i\}$, we let

$$
\epsilon_{i,t}(r_l, v) = \min_{r \in \mathcal{R}_l(r_l), S \subseteq \mathcal{S}(r)} \left\{ \sum_i v_{i,t}(r_i) - \sum_j P_j(S) \left[ f_j + \sum_{i \in I_j} [v_{i,t+1}(r_i - 1) - v_{i,t+1}(r_i)] \right] - \sum_i v_{i,t+1}(r_i) \right\}
$$

where the argument $v$ emphasizes the dependence on the given approximation. Note that if $v = \{v_{i,t}(r_i) \mid \forall t, i, r_i \in \mathcal{R}_i\}$ is feasible to $(PL)$, then $\epsilon_{i,t}(r_i, v) \geq 0$ for all $t, i$ and $r_i \in \mathcal{R}_i$. We begin with a preliminary result.

**Lemma 2.** There exists an optimal solution $\hat{v} = \{\hat{v}_{i,t}(r_i) \mid \forall t, i, r_i \in \mathcal{R}_i\}$ to $(PL)$ such that for all $t, i$ and $r_i \in \mathcal{R}_i$, we have $\epsilon_{i,t}(r_i, \hat{v}) = 0$.

**Proof.** Let $v = \{v_{i,t}(r_i) \mid \forall t, i, r_i \in \mathcal{R}_i\}$ be an optimal solution to problem $(PL)$. Let $s$ be the largest time index such that there exists a resource $l$ and $r_l \in \mathcal{R}_l$ with $\epsilon_{i,s}(r_i, v) > 0$. Since $v$ is feasible, this means that $\epsilon_{i,s}(r_i, v) = 0$ for all $t > s$, $i$ and $r_i \in \mathcal{R}_i$. We consider decreasing $v_{i,s}(r_i)$ alone by $\epsilon_{i,s}(r_i, v)$ leaving all the other components of $v$ unchanged. That is, let $\hat{v} = \{\hat{v}_{i,t}(r_i) \mid \forall t, i, r_i \in \mathcal{R}_i\}$ where

$$
\hat{v}_{i,t}(x) = \begin{cases} v_{i,t}(x) - \epsilon_{i,s}(r_i, v) & \text{if } i = l, t = s, x = r_l \\ v_{i,t}(x) & \text{otherwise.} \end{cases}
$$

Note that since $\hat{v}_{i,s}(r_i) \leq v_{i,t}(r_i)$ for all $t, i$ and $r_i \in \mathcal{R}_i$, we have $\sum_i \hat{v}_{i,1}(r_{i,1}) \leq \sum_i v_{i,1}(r_{i,1})$. Next, we show that $\hat{v}$ is feasible. Since $\hat{v}$ differs from $v$ only in one component, we only have to check those constraints where $\hat{v}_{i,s}(r_i)$ appears. Observe that $\hat{v}_{i,s}(r_i)$ appears only in the constraints for time periods $s - 1$ and $s$. For time period $s - 1$, we have

$$
\sum_j P_j(S) \left[ f_j + \sum_{i \in I_j} \hat{v}_{i,s}(r_i - 1) \right] + \sum_i \left[ 1 - \sum_{j \in J_i} P_j(S) \right] \hat{v}_{i,s}(r_i)
$$

$$
\leq \sum_j P_j(S) \left[ f_j + \sum_{i \in I_j} v_{i,s}(r_i - 1) \right] + \sum_i \left[ 1 - \sum_{j \in J_i} P_j(S) \right] v_{i,s}(r_i)
$$

$$
\leq \sum_i v_{i,s-1}(r_i)
$$

$$
= \sum_i \hat{v}_{i,s-1}(r_i)
$$

for all $r \in \mathcal{R}$ and $S \subseteq \mathcal{S}(r)$, where the first inequality follows since $\hat{v}_{i,s}(r_i) \leq v_{i,s}(r_i)$ and $\sum_{j \in J_i} P_j(S) \leq 1$, the second inequality follows from the feasibility of $v$ and the equality follows.
from (9). For time period $s$, $\hat{v}_{l,s}(r_l)$ appears only in constraints corresponding to $r \in \mathcal{R}_l(r_l)$. For $r \in \mathcal{R}_l(r_l)$, we have

$$
\sum_i \hat{v}_{l,s}(r_i) = \sum_i v_{i,s}(r_i) - \epsilon_{l,s}(r_l, v)
$$

$$
\geq \sum_j P_j(S) \left[ f_j + \sum_{i \in I_j} v_{i,s+1}(r_i - 1) - v_{i,s+1}(r_i) \right] + \sum_i v_{i,s+1}(r_i)
$$

$$
= \sum_j P_j(S) \left[ f_j + \sum_{i \in I_j} \hat{v}_{i,s+1}(r_i - 1) - \hat{v}_{i,s+1}(r_i) \right] + \sum_i \hat{v}_{i,s+1}(r_i)
$$

for all $S \subseteq \mathcal{S}(r)$, where the inequality follows from the definition of $\epsilon_{l,s}(r_l, v)$ and the last equality follows from (9). Therefore $\hat{v}$ is feasible, which implies that $\epsilon_{l,s}(r_l, \hat{v}) \geq 0$ for all $t, i$ and $r_l \in \mathcal{R}_l$. Next, we note from (9) that $\epsilon_{l,s}(r_l, \hat{v}) = 0$ for all $t > s, i$ and $r_l \in \mathcal{R}_l$. For time period $s$, since $\hat{v}_{l,s}(r_l) \leq v_{i,s}(r_l)$ and $\hat{v}_{i,s+1}(r_l) = v_{i,s+1}(r_l)$, it follows that $\epsilon_{l,s}(r_l, \hat{v}) \leq \epsilon_{l,s}(r_l, v)$. Therefore, if $\epsilon_{l,s}(r_l, v)$ was zero, then $\epsilon_{l,s}(r_l, \hat{v})$ is also zero. Moreover, $\epsilon_{l,s}(r_l, \hat{v}) = 0 < \epsilon_{l,s}(r_l, v)$.

To summarize, $\hat{v}$ is an optimal solution with $\epsilon_{l,s}(r_l, \hat{v}) = 0$ for all $t > s, i$ and $r_l \in \mathcal{R}_l$ and $\{ \epsilon_{i,s}(r_l, \hat{v}) \mid \epsilon_{i,s}(r_l, \hat{v}) > 0 \} < \{ \epsilon_{i,s}(r_l, v) \mid \epsilon_{i,s}(r_l, v) > 0 \}$. We repeat the above procedure finitely many times to obtain an optimal solution $\hat{v}$ with $\epsilon_{l,s}(r_l, \hat{v}) = 0$ for all $t \geq s, i$ and $r_l \in \mathcal{R}_l$. Repeating the entire procedure for time periods $s - 1, \ldots, 1$ completes the proof.

We are ready to prove Lemma 1. By Lemma 2, we can assume without loss of generality that the optimal solution $\hat{v} = \{ \hat{v}_{l,t}(r_l) \mid \forall t, i, r_l \in \mathcal{R}_l \}$ satisfies $\epsilon_{l,t}(r_l, \hat{v}) = 0$ for all $t, i$ and $r_l \in \mathcal{R}_l$. The proof proceeds by induction on the time periods. It is easy to see that statements hold for time period $\tau$. Assuming that the statements hold for all time periods $s > t$, we show below that the statements hold for time period $t$ as well.

**Lemma 3.** Assume that statements (i)-(iv) of Lemma 1 hold for time periods $t > s$, then statement (i) holds for time period $t$.

**Proof.** Fix a resource $l$. For $r_l > 1$, Lemma 2 implies that there exists $x \in \mathcal{R}_l(r_l - 1)$ and $S \subseteq \mathcal{S}(x)$ such that

$$
\hat{v}_{l,t}(r_l - 1) + \sum_{i \neq l} \hat{v}_{i,t}(x_i)
$$

$$
= \sum_j P_j(S) \left[ f_j + \sum_{i \neq l} \mathbb{1}_{[i \in I_j]} [\hat{v}_{i,t+1}(x_i - 1) - \hat{v}_{i,t+1}(x_i)] + \mathbb{1}_{[i \in I_j]} [\hat{v}_{i,t+1}(r_l - 1) - \hat{v}_{i,t+1}(r_l - 1)] \right]
$$

$$
\; + \hat{v}_{l,t+1}(r_l - 1) + \sum_{i \neq l} \hat{v}_{i,t+1}(x_i).
$$

(10)

Now consider the capacity vector $y$ with $y_i = x_i$ for $i \neq l$ and $y_l = r_l$. Since $x \leq y$, $\mathcal{S}(x) \subseteq \mathcal{S}(y)$
and it follows that \( S \subseteq S(y) \). Since \( \hat{v} \) is feasible, we have

\[
\hat{v}_{t,t}(r_l) + \sum_{i \neq l} \hat{v}_{i,t}(x_i)
\]

\[
\geq \sum_j P_j(S) \left[ f_j + \sum_{i \neq l} \mathbb{I}_{[s \in \mathcal{Z}_j]} [\hat{v}_{i,t+1}(x_i) - \hat{v}_{i,t+1}(x_i)] + \mathbb{I}_{[r \in \mathcal{Z}_j]} [\hat{v}_{l,t+1}(r_l) - \hat{v}_{l,t+1}(r_l)] \right]
\]

\[
+ \hat{v}_{l,t+1}(r_l) + \sum_{i \neq l} \hat{v}_{i,t+1}(x_i). \tag{11}
\]

Subtracting (10) from (11), we get

\[
\hat{v}_{t,t}(r_l) - \hat{v}_{t,t}(r_l - 1)
\]

\[
\geq \sum_j P_j(S) \mathbb{I}_{[s \in \mathcal{Z}_j]} [2\hat{v}_{l,t+1}(r_l - 1) - \hat{v}_{l,t+1}(r_l) - \hat{v}_{l,t+1}(r_l - 2)] + \hat{v}_{l,t+1}(r_l) - \hat{v}_{l,t+1}(r_l - 1)
\]

\[
\geq \hat{v}_{l,t+1}(r_l) - \hat{v}_{l,t+1}(r_l - 1)
\]

where the last inequality follows from the induction assumption that \( 2\hat{v}_{l,t+1}(r_l - 1) - \hat{v}_{l,t+1}(r_l) - \hat{v}_{l,t+1}(r_l - 2) \geq 0 \). The case \( r_l = 1 \) can be shown to hold in a similar manner. Therefore, part (ii) of Lemma 1 holds for time period \( t \).

**Lemma 4.** Assume that statements (i)-(iv) of Lemma 1 hold for time periods \( t > s \), then statement (ii) holds for time period \( t \).

**Proof.** For \( r_l \in \mathcal{R}_l \setminus \{0, r_l^1\} \), Lemma 2 implies that there exists \( x \in \mathcal{R}_l(r_l + 1) \) and \( S \subseteq S(x) \) such that

\[
\hat{v}_{t,t}(r_l + 1) + \sum_{i \neq l} \hat{v}_{i,t}(x_i)
\]

\[
= \sum_j P_j(S) \left[ f_j + \sum_{i \neq l} \mathbb{I}_{[s \in \mathcal{Z}_j]} [\hat{v}_{i,t+1}(x_i) - \hat{v}_{i,t+1}(x_i)] + \mathbb{I}_{[r \in \mathcal{Z}_j]} [\hat{v}_{l,t+1}(r_l) - \hat{v}_{l,t+1}(r_l + 1)] \right]
\]

\[
+ \hat{v}_{l,t+1}(r_l + 1) + \sum_{i \neq l} \hat{v}_{i,t+1}(x_i). \tag{12}
\]

Now consider the capacity vector \( y \) with \( y_i = x_i \) for \( i \neq l \) and \( y_l = r_l \). Since, \( r_l \geq 1 \), \( S(y) = \{ j \mid \mathbb{I}_{[s \in \mathcal{Z}_j]} \leq y_i, i \neq l; \mathbb{I}_{[r \in \mathcal{Z}_j]} \leq r_l \} = \{ j \mid \mathbb{I}_{[s \in \mathcal{Z}_j]} \leq x_i, i \neq l; \mathbb{I}_{[r \in \mathcal{Z}_j]} \leq r_l + 1 \} = S(x) \). Therefore, \( S \subseteq S(y) \) and since \( \hat{v} \) is feasible

\[
\hat{v}_{t,t}(r_l) + \sum_{i \neq l} \hat{v}_{i,t}(x_i)
\]

\[
\geq \sum_j P_j(S) \left[ f_j + \sum_{i \neq l} \mathbb{I}_{[s \in \mathcal{Z}_j]} [\hat{v}_{i,t+1}(x_i) - \hat{v}_{i,t+1}(x_i)] + \mathbb{I}_{[r \in \mathcal{Z}_j]} [\hat{v}_{l,t+1}(r_l - 1) - \hat{v}_{l,t+1}(r_l)] \right]
\]

\[
+ \hat{v}_{l,t+1}(r_l) + \sum_{i \neq l} \hat{v}_{i,t+1}(x_i). \tag{13}
\]
Subtracting (13) from (12), we get
\[
\hat{v}_{i,t}(r_i + 1) - \hat{v}_{i,t}(r_i) \\
\leq \sum_j P_j(S) \| \hat{z}_{i,t+1}(r_i) - \hat{v}_{i,t+1}(r_i - 1) - \hat{v}_{i,t+1}(r_i + 1) + \hat{v}_{i,t+1}(r_i + 1) - \hat{v}_{i,t+1}(r_i) \\
\leq 2\hat{v}_{i,t+1}(r_i) - \hat{v}_{i,t+1}(r_i - 1) - \hat{v}_{i,t+1}(r_i + 1) + \hat{v}_{i,t+1}(r_i + 1) - \hat{v}_{i,t+1}(r_i) \\
= \hat{v}_{i,t+1}(r_i) - \hat{v}_{i,t+1}(r_i - 1) \\
\leq \hat{v}_{i,t}(r_i) - \hat{v}_{i,t}(r_i - 1),
\]
where the second inequality follows from the induction assumption that \(\hat{v}_{i,t+1}(r_i) - \hat{v}_{i,t+1}(r_i - 1) \geq \hat{v}_{i,t+1}(r_i) - \hat{v}_{i,t+1}(r_i)\) and the fact that \(\sum_j P_j(S) \| \leq 1\). The last inequality follows from Lemma 3. Therefore, part (iii) of Lemma 1 holds for time period \(t\).

**Lemma 5.** Assume that statements (ii)-(iv) of Lemma 1 hold for time periods \(t > s\), then statement (iii) holds for time period \(t\).

**Proof.** By the induction assumption, \(\hat{v}_{i,t+1}(r_i) \geq \hat{v}_{i,t+1}(r_i - 1)\) for \(r \in \mathcal{R} \setminus \{0\}\). The result now follows from Lemma 3.

**Lemma 6.** Assume that statements (ii)-(v) of Lemma 1 hold for time periods \(t > s\), then statement (iv) holds for time period \(t\).

**Proof.** We first show that \(\hat{v}_{i,t}(0) \geq \hat{v}_{i,t+1}(0)\). Suppose there exists \(l\) with \(\hat{v}_{l,t}(0) < \hat{v}_{l,t+1}(0)\). Since \(\hat{v}\) is feasible, it satisfies constraint (7) for the state vector \(r = 0\) and \(S = \emptyset\). That is
\[
\hat{v}_{l,t}(0) + \sum_{i \neq l} \hat{v}_{i,t}(0) \geq \hat{v}_{l,t+1}(0) + \sum_{i \neq l} \hat{v}_{i,t+1}(0),
\]
where we use \(P_j(0) = 0\) for all \(j\). This implies there exists \(k\) with \(\hat{v}_{k,t}(0) > \hat{v}_{k,t+1}(0)\). Letting \(\delta = \min\{\hat{v}_{l,t+1}(0) - \hat{v}_{l,t}(0), \hat{v}_{k,t}(0) - \hat{v}_{k,t+1}(0)\} > 0\) and
\[
\hat{v}_{i,s}(x) = \begin{cases} 
\hat{v}_{i,s}(x) + \delta & \text{if } i = l, s = t, x \in \mathcal{R}_l \\
\hat{v}_{i,s}(x) - \delta & \text{if } i = k, s = t, x \in \mathcal{R}_k \\
\hat{v}_{i,s}(x) & \text{otherwise,}
\end{cases}
\]
it can be verified that \(\hat{v}\) is also an optimal solution to \((PL)\). Moreover, since \(\hat{v}\) satisfies properties (i)-(iii) for time periods \(s \geq t\), so does \(\hat{v}\). If \(\hat{v}_{l,t}(0) < \hat{v}_{l,t+1}(0)\), then by repeating the above arguments, there exists a resource \(k'\) with \(\hat{v}_{k',t}(0) > \hat{v}_{k',t+1}(0)\). Repeating the above procedure finitely many times, we obtain an optimal solution \(\hat{v}\) with \(\hat{v}_{i,0}(0) \geq \hat{v}_{i,1}(0)\) for all \(i\).

Now assume that \(\hat{v}_{i,t}(r_i - 1) \geq \hat{v}_{i,t+1}(r_i - 1)\). Lemma 3 implies that \(\hat{v}_{i,t}(r_i) \geq \hat{v}_{i,t+1}(r_i)\).